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A Vectorizable Random Lattice

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We describe a family of random lattices in which the connectivity is determined by the Voronoi construction while the vectorizability is not lost. We can continuously vary the degree of randomness so in a certain limit a regular lattice is recovered. Several statistical properties of the cells and bonds of these lattices are measured. We also study anisotropy effects on the numerical solution of the Laplace equation for varying degrees of randomness.

KEY WORDS: Random lattices; vector algorithms.

1. INTRODUCTION

The introduction of a spatiotemporal lattice for the discretization of physical systems is a widely used tool as a means to regularize field theories or just to numerically solve some continuous equation. The underlying assumption in doing so, i.e., the existence of a well-defined continuum limit, may be broken by the regularity of the lattice, which reduces the symmetries of the physical system to a discrete set of transformations. We will losely refer to this as the *anisotropy* of the lattice.

A way out of this problem has been proposed by Christ *et al.*,[1] who describe how to discretize a system by making use of a *random lattice* (RL), that is, one for which the sites are a set of randomly chosen points with uniform distribution. In this way the spatiotemporal symmetries of the continuum system are recovered after averaging over lattice realizations. The number of lattice points on a given volume V is then a random variable with Poisson distribution. We can then call these lattices *Poisson random lattices* (PRL).

From the numerical point of view this approach is disadvantageous because the regularities of a lattice can often be employed to substantially

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improve the performance of the simulation. No such regularities exist on the RL which have been defined so far, so it has not been possible to perform large-scale simulations on them. The reason is of course the lack of parallelism produced by the fact that each site has different characteristics. The number and relative location of the nearest neighbors of a given site are themselves. In fact, for the PRL the number of nearest neighbors of a site is not even bounded, basically because any spatial ditribution of points is possible.

Our goal here is to define a new type of random lattice that does not introduce any anisotropy into the system, but it is still vectorizable or parallelizable. For this to be achieved we have to retain a certain degree of regularity that, being enough to enable vectorization, does not appear in the numerical solution of a physical system. The degree of randomness is a tunable parameter in our family of lattices. When this randomness is completely eliminated a regular lattice is recovered and anisotropic effects show up again.

In Section 2 we generally review the basics of random lattices and in Section 3 we introduce the vectorizable random lattice (VRL). Some statistic properties of the VRL have been measured and the results are described in Section 4. Section 5 contains a quantitative analysis of anisotropy effects on the numerical solution of the Laplace equation for varying degrees of randomness, while Section 6 contains our conclusions.

2. RANDOM LATTICES

Given a set of arbitrarily distributed points in d-dimensional space, a natural prescription for interconnecting them is the following[2]:

First determine the Voronoi cells associated to these sites, that is, the open subsets of space points that are nearer to a given site than to any other. Neighboring cells will then share a face. Each pair of sites whose cells share a face are defined to be connected by a link. This link is perpendicular to this face, but does not always intersect it. An alternative and equivalent definition is the one due to Christ *et al.*[1]: Any set of $d+1$ such points is said to form a *d-simplex* if the d-dimensional hypersphere in which they are inscribed (its circumscribed hypersphere) contains no other point inside. Any two points of a d-simplex will be connected by a link. Such simplices are nonoverlapping and their superposition covers the entire space.[1] This construction we will call the *direct* lattice. In two dimensions the simplices are triangles, in three dimensions they are tetrahedra, and so on.

The direct lattice and the Voronoi lattice are dual to each other. The Voronoi cells are delimited by $(d-1)$-dimensional hyperplanes (faces)

which are shared by two neighboring cells. These hyperplanes are delimited by $(d-2)$-dimensional hyperplanes and so on. The sites of the Voronoi lattice are the zero-dimensional intersections of $d+1$ cells. There is a one-to-one correspondence between sites of the direct lattice and Voronoi cells, in the same way as between sites of the Voronoi lattice and d-simplices. In general there is a mapping between $(d-d^*)$-dimensional objects in the direct lattice and d^*-dimensional objects in the dual lattice. We see the problem of finding the connectivities of the direct lattice is fully equivalent to that of finding the associated Voronoi cell construction, for any given set of points.

Once the lattice connectivities have been found, one has to describe the particular physical system of interest, writing down its constitutive equations in terms of discrete variables defined on nodes, links, plaquettes, etc. It is not always clear how to do this, and for certain system which do not have a continuum counterpart there is indeed a certain ambiguity in this step. But in some cases where the continuum equations are known, a presciption can be given to write down the equations of motion in terms of the discrete versions of the (in general differential) operators of the theory. A very comprehensive review of the way in which this is done can be found in the work of Itzykson.[3]

In the specific case of solving the Laplace equation on a random lattice one obtains a set of linear equations of the form

$$-(\Delta\phi)_i = \sum_{j_{(i)}} (\phi_i - \phi_j)\omega_{ij} = 0 \tag{1}$$

where $j_{(i)}$ are the neighbors of i and ω_{ij} are coupling constants that are calculated in terms of the geometrical properties of the (Voronoi) cells i and j,[3]

$$\omega_{ij} = \frac{1}{\sigma_i}\frac{\sigma_{ij}}{l_{ij}} \tag{2}$$

where σ_i is the area of the Voronoi cell i, σ_{ij} is the length of the common face between cells i and j, and l_{ij} is the distance between sites i and j.

3. THE VECTORIZABLE RANDOM LATTICE (VRL)

From the point of view of algorithmic parallelism, two conditions are desirable in the case of lattice systems:

(a) The possibility to label the sites of the lattice by a n-tuple of integers.

(b) The existence of a regularity in the neighborhoods of the sites in the sense that, given any site, one has a simple rule to know which are its neighbors.

Neither of these is satisfied for a random lattice of the type described above. In fact, we have unsuccessfully attempted to define at least a regular labeling in the case of two-dimensional Poissonian random lattices.

To overcome these difficulties, we introduce a lattice in which these two conditions are met by construction, but one in which there is a certain degree of disorder which should avoid anisotropy. The only step we have to change to get this is the way in which the lattice sites sites are chosen in space, in the following way:

Let us describe the idea for the case of a two-dimensional lattice. We first define a regular (e.g., square) lattice, which we call the reference lattice. The Voronoi cells dual to its sites are in this case squares that do not overlap and tile the entire space. We call them *reference cells.*

Next we randomly pick in each cell a point with uniform distribution. These will be the sites of our direct lattice. The spatial distribution of these points is homogeneous, i.e., $P(x) = \text{const}$. The last step is to define the connectivity of each point according to the usual prescription,[1] i.e., take three

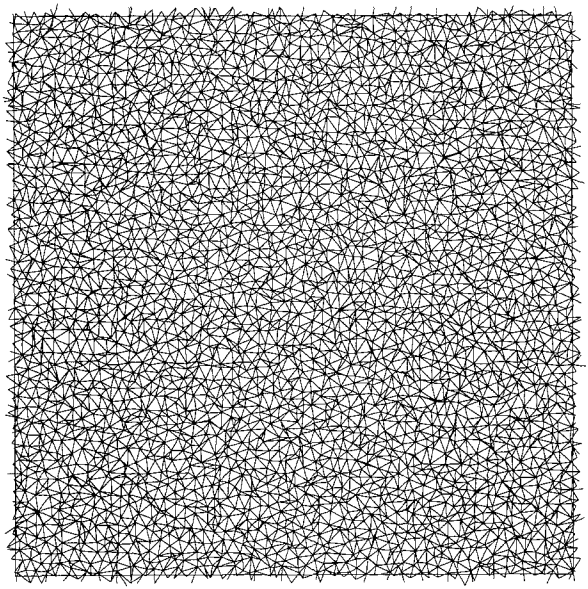

Fig. 1. An example of the vectorizable random lattice with 61×61 sites. Periodic boundary conditions were used. Notice the regularity in triangle size as compared to the Poissonian random lattice (Fig. 2).

points, and check whether their circumscribing circle (the smallest circle in which they are all located) contains no other point inside. If this is so, these three points will form a triangle and they will be connected to each other.

In Fig. 1 we show an example of a lattice which has been constructed according to this prescription. We can see that the appearance is so random that one is not able to recognize the underlying regularity with the naked eye. We also include for comparison an example of the Poissonian random lattice with the same number of points (Fig. 2).

For the vectorizable random lattice, condition (a) is met by construction in a trivial way. The second condition is not fully satisfied, because the connectivities of different sites will still be random, but now we have the advantage that their number is bounded. Owing to the fact that each reference cell has to contain one and only one point, it is easily seen that not any pair of points can be connected. Regardless of the actual positions that the sites have inside their reference cells, if we take two of them which are sufficiently far apart, then any circumscribing circle in which these lie will also contain entire reference cells inside and so necessarily their associated sites.

Let us suppose that site x_0 belongs to the cell c_0 of the reference lattice. Then the only sites x_j that can be connected to x_0 are those

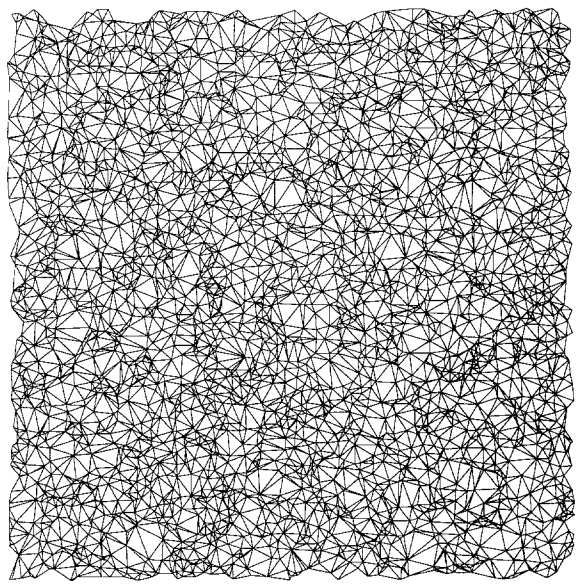

Fig. 2. An example of the Poissonian random lattice with 3600 sites. Lattice sites are in this case randomly located on the plane.

belonging to a certain cluster of reference cells around c_0. We call this cluster the *potential neighbors* of x_0 and it can be easily determined for each type of lattice.

In Fig. 3 we show the cluster of potential neighbors around a given site of the square lattice. The circumscribing circles pass through three points, one of which is the central one x_0. For sites x_k outside the cluster it is easily seen that there exists no circle passing through x_0 and x_k that does not contain at least another point *inside*.

Since we do not know in advance which ones of the potential neighbors of a given site will be connected to it (this will depend on the relative location of the sites within the cells), we can proceed as if all of them were connected, because their number is limited. Algorithmically speaking, each time an operation requires the use of the neighbors of a site, we will go through all sites which are potential neighbors, but of course only those which are really connected will have a nonzero coupling.

In a square lattice the potential neighbors belong to a polygon which can be inscribed in a 7×7 square around the central site (Fig. 3). In the triangular lattice, the potential neighbors are all those which are not farthest than third nearest neighbors.

Essential to allow for vectorization or parallelization of the system is that the number of potential neighbors of a given site is limited and their location is known, even when not all of them will be connected to it. While

		4.12×10^{-5}	1.45×10^{-5}	4.12×10^{-5}		
	8.08×10^{-5}	1.56×10^{-2}	4.06×10^{-2}	1.56×10^{-2}	8.08×10^{-5}	
4.12×10^{-5}	1.56×10^{-2}	4.93×10^{-1}	9.36×10^{-1}	4.93×10^{-1}	1.56×10^{-2}	4.12×10^{-5}
1.45×10^{-5}	4.06×10^{-2}	9.36×10^{-1}		9.36×10^{-1}	4.06×10^{-2}	1.45×10^{-5}
4.12×10^{-5}	1.56×10^{-2}	4.93×10^{-1}	9.36×10^{-1}	4.93×10^{-1}	1.56×10^{-2}	4.12×10^{-5}
	8.08×10^{-5}	1.56×10^{-2}	4.06×10^{-2}	1.56×10^{-2}	8.08×10^{-5}	
		4.12×10^{-5}	1.45×10^{-5}	4.12×10^{-5}		

Fig. 3. Cluster of potential neighbors (reference cells that can be connected to the central one). Cells other than these cannot be connected to the central one. The numbers indicate the probability for these cells to be connected, as estimated numerically from 10^6 independent realizations. It is apparent here that outer shell of potential neighbors has a very low probability of connection.

the mean number of neighbors per site is six, the number of potential neighbors one has to take into account can be still quite high, 36 in the square lattice and 42 in the triangular lattice. Now we describe a further modification that improves the performance of the simulations by discarding from the cluster of potential neighbors those sites with very low probability of connection.

We have measured the probability for a potential neighbor to be connected, by averaging over 10^6 sites, and found the results shown in Fig. 3. We see that there are sites with a very low probability, namely of order 10^{-5}, to be connected. These are mostly outside the 5×5 square around the central site. On the other hand, the innermost sites have connection probabilities of order 10^{-2} or higher, so in this lattice there exists a natural division allowing the reduction of the number of potential neighbors. To implement this reduction of the potential neighborhood, we consider this outer shell of highly improbable connections as "forbidden" and accept only those lattices with no forbidden connections. In this work we have taken as "allowed" connections only those to cells contained in the 5×5 cluster around a site, with the exception of its corners [$(\pm 2, \pm 2)$ in local coordinates], so reducing the potential neighborhood to 20 sites.

In practice we calculate the connectivities of the sites according to the Voronoi prescription. Next all sites with forbidden conections are removed and again a new random point is selected in their cells. The connectivities are then recalculated. This procedure is repeated until no forbidden connections appear. We do not expect the modifications introduced in this way to be relevant, because the appearance of forbidden connections is itself very rare. Typically one or two sites will have to be corrected in a lattice of 10^4 sites.

On these lattices we used the conjugate gradient method to solve numerically the Laplace equation. For comparison the same was implemented on the regular square lattice, in which case one can substantially reduce the number of code lines. We give for comparison the cpu time per sweep for both, running on one processor of the Cray Y-MP:

For the case of the regular square latice of size 101×101 the performance is 2.4×10^{-3} sec/sweep, while for the random lattice of the same size we need 6.4×10^{-3} sec/sweep. We see that there is a downgrading by a factor of about three, due to the greater number of neighbors one has to take into account in each loop. But both codes vectorize.

Finally let us mention that the way in which these lattices are defined allows us to tune its degree of randomness. One possible and simple way of doing so is, for example, reducing the size of the regions in which the points are chosen randomly by scaling them down by a factor A. We choose then these regions to be squares of length A in the center of the

reference cells, of length unity. This sites of the random lattice will then be confined to be enclosed within these reduced regions. We have in this way a family of random lattices characterized by a number A, $0 \leqslant A \leqslant 1$, such that in the limit $A = 1$ maximal randomness is obtained, while for $A = 0$ we recover the regular reference lattice.

4. STATISTICAL PROPERTIES OF THE VRL

We have measured several statistical properties of these lattices for $A = 1$. In Table I we can see the distribution of the number of neighbors n averaged over 10^6 lattices sites. The maximum number of neighbors we have found is 12, with one event out of 10^6. It is reasonable that compared to the same distribution in the Poissonian random lattice,[4] in this case the tail of high n is strongly suppressed due to the existence for an upper bound.

The area (Ω) distribution of the Voronoi polygons has also been numerically estimated, and it is found to be approximately symmetric around $\Omega = 1$ (Fig. 4). Comparing to the case of Poissonian RL,[3,4] we find that Ω has less fluctuations in this case, and its distribution is more symmetric. Very large as well as very small areas are highly improbable for our lattices. The same situation is found when polygon perimeters (Fig. 4)

Table I. Distribution of the Number n of Neighbors in the VRL, Estimated from 10^6 Statistically Independent Cells

n	Probability
1	0.00
2	0.00
3	1.650×10^{-3}
4	5.615×10^{-2}
5	2.671×10^{-1}
6	3.775×10^{-1}
7	2.232×10^{-1}
8	6.415×10^{-2}
9	9.301×10^{-3}
10	8.270×10^{-4}
11	4.700×10^{-5}
12	1.000×10^{-6}
13	0.00
14	0.00

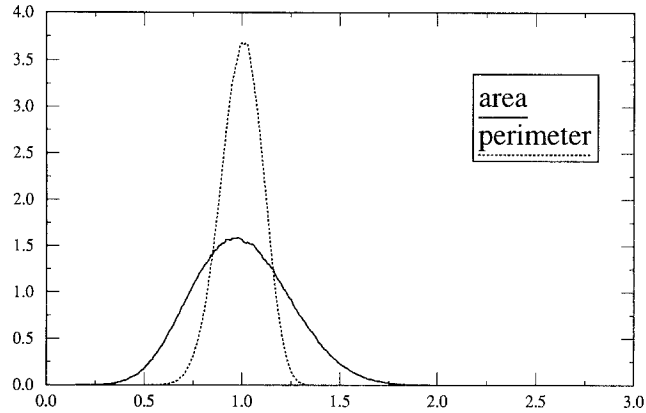

Fig. 4. Distribution of areas and perimeters of Voronoi cells in the vectorizable random lattice, estimated from 10^6 independent neighborhoods. The perimeter scale has been reduced by a factor 1/4.

or distances to neighbors (Fig. 5) are measured. Their distributions are in this case very highly peaked around the average value.

The sites in these lattices are kept apart by the constraint that no two sites can lie within the same reference cell. The maximum number of points that can be arbitrarily close in this case is four. It is also reasonable to expect that the problems found in Poissonian RL,[3] which are associated with the existence of points extremely close to each other, will be less severe in the VRL due to the suppression of very close neighbors.

We have also measured the average properties of the links of this lattice. The first such magnitude is the probability $P_c(r, \theta)$ that a site

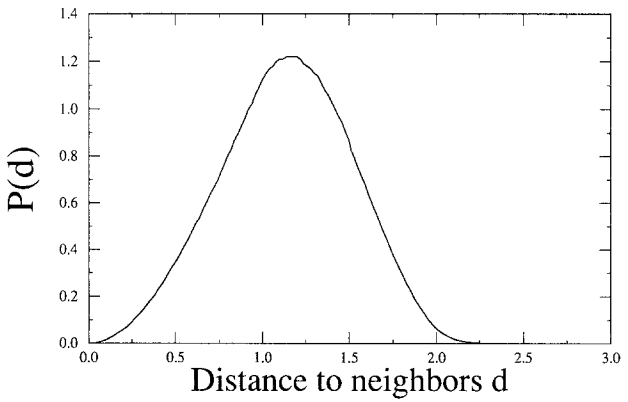

Fig. 5. Distribution of distances between the central site and the neighbors which are connected in the vectorizable random lattice, estimated from 10^6 independent neighborhoods.

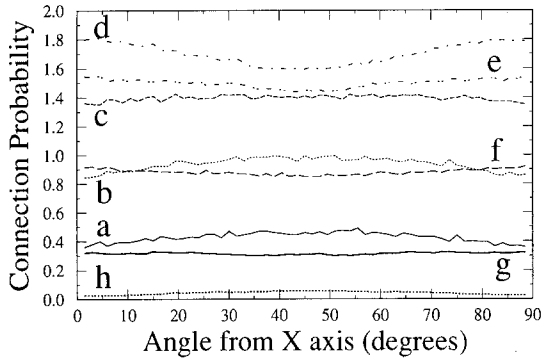

Fig. 6. Probability $P_c(r, \theta)$ per unit area to find a (connected) neighbor at the position (r, θ) from a site, in polar coordinates, estimated by averaging over 10^6 independent cells of the vectorizable random lattice. We have plotted $P_c(r, \theta)$ as a function of θ for several values of r: (a) $r \leqslant 0.3$, (b) $0.3 \leqslant r \leqslant 0.6$, (c) $0.6 \leqslant r \leqslant 0.9$, (d) $0.9 \leqslant r \leqslant 1.2$, (e) $1.2 \leqslant r \leqslant 1.5$, (f) $1.5 \leqslant r \leqslant 1.8$, (g) $1.8 \leqslant r \leqslant 2.1$, (h) $2.1 \leqslant r \leqslant 2.4$. Due to the fourfold symmetry of the problem we only show the sector $0 \leqslant \theta \leqslant \pi/2$.

located at (0, 0) has a neighbor at the point (r, θ) in polar coordinates. If the local properties of the lattice were fully isotropic, no θ dependence would be observable. In Fig. 6 a plot is shown of this P_c as a function of θ for several values of r. We only show the $0 \leqslant \theta \leqslant \pi/2$ sector. These values were obtained by making a histogram in two variables and averaging over 10^6 independent cells.

It is evident from this plot that the spatial distribution of neighbors still has some mild anisotropy. This is of course not surprising, because the reference lattice is anisotropic, so even though the distribution of lattice

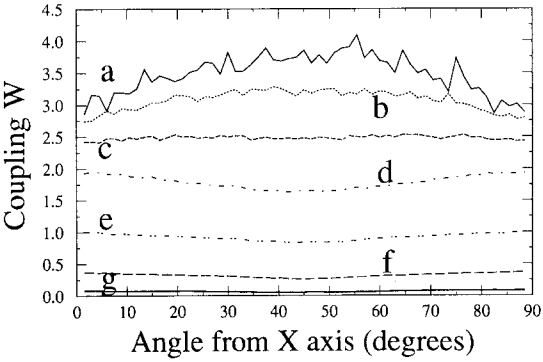

Fig. 7. The average coupling $w(r, \theta)$ corresponding to the discrete Laplace equation, plotted in the same way as in Fig. 6. The coupling shows a monotonic decrease with the distance between sites due to the factor $1/l_{ij}$ in the definition of w_{ij} [Eq. (2)].

points $P(x)$ is itself homogeneous in space, the combined probability $P(x_1, x_2)$ has a weak anisotropic bias.

A similar magnitude one can measure is the mean value of the coupling $\omega_{ij}(r, \theta)$ for the case of the discretized Laplace equation [Eq. (1)]. This is shown in Fig. 7, and similar effects are apparent there.

Nevertheless, the point is whether this anisotropy has a physically observable effect. We will later see that it is so mild that it does not show up in the numerical solution of the Laplace equation.

5. GLOBAL ISOTROPY

In this section we report our measurements of the anisotropic effects on the numerical solution of the Laplace equation

$$\Delta\phi = \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right\} \phi = 0 \tag{3}$$

for varying degrees of randomness. We first find numerically the potential of a charge on a two-dimensional space with fixed boundary conditions on a circular border. After averaging over 200 lattice realizations we get

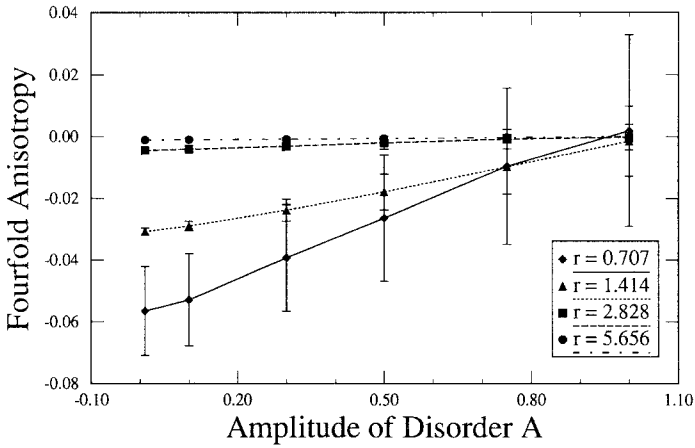

Fig. 8. Reduced fourth-order coefficients in the cosine transform (for its θ dependence) of Laplacian field on the VRL (the boundary conditions are constant on a circle with $r = 50$), as a function of the "disorder parameter" A and for several values of the distance to the origin. The field was averaged over 200 random lattices before calculating its cosine transform. For $A = 0$ (regular square lattice) there is a strong anisotropic effect at small distances from the central charge. For $A = 1$ (maximally disordered VRL), this effect is almost zero at all distances.

$\bar{\phi}(r, \theta)$. The r values were discretized in the form $r_n = 2^{(n-2)/2}$ so as to get equally spaced values for $\log r$. The θ values were also isospaced between 0 and 2π.

Since for each lattice the field is only known on the sites, we had to interpolate to find its values on arbitrary points. This we did by identifying the triangle to which this point belongs and interpolating the values of ϕ inside by a plane. In this way we obtained, for each lattice, the values of ϕ on concentric circles with radii r_n.

For each value of r_n we calculated the cosine transform of $\bar{\phi}(r, \theta)$ to get the Fourier coefficients $C_k(n)$. The relevant magnitude, in this case the "fourfold anisotropy," is C_4/C_0, which depends on r (the distance from the center) and on A (the degree of anisotropy of the lattice).

In Fig. 8 we show the results for this magnitude. For the square lattice ($A = 0$) we see that at low distances a certain anisotropy is present, but it decays as the distance from the center is increased. On the other hand, in the random lattice limit ($A = 1$) the anisotropy is almost zero at any distance from the origin.

6. CONCLUSIONS

We have defined a new type of random lattice on which it is possible to attain algorithmic parallelism, so enabling vectorization or parallelization. The advantage from the point of view of computation is that large-scale simulations on random lattices will be more feasible. The random lattices previously defined did not allow for any degree of parallelization, so the size of the systems which could be explored was limited.

The idea is to define the sites of the lattice by randomly picking points in space, but only one in each square of a regular lattice, with homogeneous distribution. The connectivities are determined by looking at the Voronoi cells, as described, for example, in ref. 2, so the appearance of the lattice is fully random, but vectorizability is not lost.

Several statistical properties of the cells and bonds of these lattices were measured. A very mild anisotropy in their local properties was revealed, but it does not seem to show up in the solution of the Laplace equation on these lattices. Of course this does not mean that this lattice will be fully isotropic for any physical system. It could be that for some other model which is more sensitive to small amounts of anisotropy, these lattices may still have some anisotropic effects.

One such example could be the noise-reduced version of DLA,[5] which is known to be very sensitively affected by lattice effects. We are presently investigating this case.

The idea enabling the definition of these vectorizable random lattices can be applied for any underlying regular or irregular reference lattice in any dimensions. In general terms one should determine for the reference lattice of interest the associated Voronoi cells and in each of them pick a point at random with uniform distribution.

In this way further generalization is possible: One could take a vectorizable random lattice as a reference lattice and in each Voronoi cell define a lattice site as described. This would give rise to a "second-generation" vectorizable random lattice with less memory of the first regular lattice, and so on.

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